Microscopic model of financial markets based on belief propagation

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Abstract

We present a simple microscopic model of financial markets based on belief propagation in order to simulate the dynamics of the stock markets. A two-dimensional small-world communication structure is introduced in our model and the beliefs of market leaders spread on the network which results in the herd behaviors of traders. Most of the stylized aspects of the financial market time series, including multifractal property, are reproduced by the model. A direct comparison is made with the daily closures of the Shenzhen composite index.

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1. Introduction

During the last years there has been considerable interest in applications of statistical physics to financial market dynamics in order to analyze and model financial time series [1]. A variety of models have been proposed over the last few years to study financial market dynamics. Based on the competition between supply

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and demand, the effort is to model the main observed “stylized facts”: fat-tailed distribution of returns, volatility clustering, multifractality [2–4]. The examples of the models include the “minority games” models [5], percolation models [6,7], spin models [8], “two-type trader” models [9], cellular automata model [10] and other microscopic models [11–14].

The heavy tails observed in the distribution of financial returns correspond to large fluctuations in prices. It is believed that the significant fluctuation in the demand and supply of market comes from the herding and imitation behaviors of market traders. For example, when a market is strongly influenced by political decisions or liquidity of a stock is limited, the only strategy to win is to search for ‘well informed’ investors to join their team or if it is impossible to follow their decisions.

In this paper, we present a model for opinion cluster growth and information dispersal by investors in the financial market. With the help of the information communication network, traders aggregate in the vicinities of the information sources which may be influential traders or market leaders. We propose a simple iteration algorithm to control the spread progress of beliefs of market leaders, which is inspired by the work on spreading activation networks [15,16], diffusion kernels [17–19], and recent work on semi-supervised learning and clustering [20–22]. We firstly define a weighted network which represents the communication structure between traders and assign a belief score to every ‘well informed’ investor (market leader). The leaders act as source nodes that continually pump their belief scores to the remaining traders via the weighted network, and the remaining traders further spread the belief scores they received to their neighbors. This spreading process is repeated, and the traders make decision according to the amounts of belief score they received. The results obtained by the simulations are then compared with the time series of daily closures of the Shenzhen Stock Exchange (SZSE) composite index over a period of about 8 years. The comparison shows the validity of the model.

The plan of the paper is as follows. In Section 2 we present our model and prove the convergence of the iteration algorithm. In Section 3 the numerical results of the model are shown and the comparisons with daily closures of the SZSE composite index are made. We discuss our work and draw some conclusions in Section 4.

2. The model

In real markets, agents may form groups of various sizes, which then may share information and act in coordination. For example, a group of investors participate in a mutual fund. To capture such effects, we need to introduce an additional ingredient, namely, the communication structure between agents. One solution would be to specify a fixed trading-group structure and then proceed to study the resulting aggregate fluctuations.

Small-world networks are well suited to study properties of physical systems with underlying networks ranging from regular lattices to random graphs [23], by changing a single parameter. The small-world network model was proposed by
Watts and Strogatz [24] for the modeling of social networks. The model is based on a locally highly connected regular lattice, in which a fraction $p$ of the links between nearest-neighbor sites are randomly replaced by new random links, thus creating long-range “shortcuts”.

In the present work, we use two-dimensional small-world as the communication structure between traders. The building progress of the two-dimensional small-world network is as follows. The network $G = (V, E)$ is built on the basis of a square lattice of size $L \times L$ with periodic boundary conditions where $V$ is the set of vertices and $E$ is the set of edges. Each site in the lattice is connected to its four nearest-neighbor sites. Then, some long-range connections, denoting a long-range jump, are added to the regular lattice randomly. The ratio between the number of long-range connections and that of short-range ones is $f$, which is a predetermined parameter.

Based on the small-world network as created above, we describe how the information spreads and traders aggregate in the financial market. At the beginning before each deal, we select $N_s$ traders randomly in the network as leaders of the market who can be seen as financial information sources. They may be professional investors or those who know some low-downs. Each of these leaders will have a belief of the market’s trend and choose buy or sell according to the information he/she holds. Other nodes in the network are set inactive. The beliefs of those market leaders spread on the network and those who are influenced by the leaders will be set active. The beliefs decrease in the propagation progress and those in the vicinities of the leaders will gain higher beliefs than those far away from the leaders on the small-world network. The spread rate of information is controlled by the market state: in the stagnancy period the information will spread more slowly and affect fewer people than that in the boom period.

We assume that the market contains $N, N = L \times L$ agents. The state of agent $i$ is represented by $s_i = \{0, +1, -1\}$ corresponding to an inactive state [waiting ($s_i = 0$)] and two active states [either buying ($s_i = +1$) or selling ($s_i = -1$)]. Initially, all agents are inactive ($s_i = 0, \forall i$). Let $\mathcal{F}$ denote the set of $N \times 2$ matrices with nonnegative entries. A matrix $F = [F_1^T, \ldots, F_n^T]^T \in \mathcal{F}$ is defined as a belief matrix. The first and second columns of $F$ correspond to the beliefs received by traders from the leaders choosing buy strategy and leaders choosing sell strategy, respectively. Define an $N \times 2$ matrix $Y \in \mathcal{F}$ with $Y_{i0} = 1$ if trader $i$ is a leader who will choose buy strategy and $Y_{i1} = 1$ if trader $i$ is a leader who will choose sell strategy in the coming deal. The herding of agents evolves dynamically in the following way:

**Step 1:** Form the trust matrix $T$ defined by $T_{ij} = T_{ji} = u_{ij}$ if there is an edge $e(i, j) \in E$ or else $T_{ij} = 0$, $u_{ij}$ is sampled from the uniform distribution in the interval $[0, 1]$. $T_{ij}$ represents the degree to which trader $i$ trusts trader $j$, i.e., the higher $T_{ij}$ is, the more trader $i$ agrees with trader $j$’s opinion.

**Step 2:** Compute the matrix $S$ defined by $S = D^{-1/2}TD^{-1/2}$ in which $D$ is the diagonal matrix with $D_{ii} = \sum_{j=1}^{N} T_{ij}$.

**Step 3:** $N_s$ traders are selected randomly as leaders of markets, who can be seen as active information sources. Each trader will show preference for buying or selling with equal probability. $Y$ is generated according to the leaders’ choices. $N_s$ is
sampled from the distribution $\rho(N_s)$ which is a uniform distribution in the interval $[1, N_{s\text{max}}]$.

Step 4: Iterate $F(t + 1) = \alpha SF(t) + (1 - \alpha)Y$, where $\alpha$ is a parameter in $(0, 1)$. This iteration repeats $\gamma$ times which is the propagation time of the beliefs of leaders. The parameter $\alpha \in (0, 1)$ is used to control the beliefs received from neighbors who are not leaders.

Step 5: All the agents on the information network become active instantly. $F$ contains all the accumulative beliefs of traders received from leaders. If $F_{i0} > F_{i1}$ and $F_{i0} > \beta$ then agent $i$ chooses the strategy $s_i = +1$; If $F_{i0} < F_{i1}$ and $F_{i1} > \beta$ then agent $i$ chooses the strategy $s_i = -1$; others remain inactive. $\beta$ is a belief threshold and those who receive belief scores higher than $\beta$ will participate in the coming deal. The aggregate state of the system $x = \sum_{i=1,N} s_i$ is computed. After that the clusters on the network are broken up and their states are reset $s_i = 0, \forall i$. The number of propagation time $\gamma$ will be adjusted. The increment is a fraction $p$ of $x$.

The normalization in the second step is necessary to guarantee the algorithm’s convergence. Refs. [21,22] show that the sequence $\{F(t)\}$ converges and the limit $F^*$ is $(1 - \alpha)(I - \alpha S)^{-1}Y$. Without loss of generality, suppose $F(0) = Y$. By the iteration equation $F(t + 1) = \alpha SF(t) + (1 - \alpha)Y$ used in the algorithm, we have

$$F(t) = (\alpha S)^{t-1}Y + (1 - \alpha) \sum_{i=0}^{t-1} (\alpha S)^i Y.$$ (1)

Since $0 < \alpha < 1$ and the eigenvalues of $S$ are in $[-1, 1]$,

$$\lim_{t \to \infty} (\alpha S)^{t-1} = 0$$ (2)

and

$$\lim_{t \to \infty} \sum_{i=0}^{t-1} (\alpha S)^i = (I - \alpha S)^{-1}. (3)$$

Hence

$$F^* = \lim_{t \to \infty} F(t) = (1 - \alpha)(I - \alpha S)^{-1}Y.$$ (4)

Now we can compute $F^*$ directly without iterations. This also shows that the iteration result does not depend on the initial value for the iteration. In addition, it is worth noticing that $(I - \alpha S)^{-1}$ is in fact a graph or diffusion kernel [21,22].

The evolution of the system is characterized by a succession of discrete events $x_1, x_2, \ldots$, which correspond to the difference between the supply and the demand of every deal.

3. Numerical results

We have presented a basic framework of the microscopic market model based on belief propagation in the above section, and it is easy to examine its dynamical behavior by numerical simulation.
In this context, the dynamics of the price index $p(t)$ can be easily derived if we assume that the index variation is proportional to the difference between demand and supply:

$$\frac{dp(t)}{dt} \propto x(t)p(t), \quad (5)$$

$$x(t) = \sum_{i=1}^{N} s_i(t) \quad (6)$$

We follow the simple update rule for the price index $p(t)$ discussed in Ref. [25] in which the price $p(t)$ evolves as $p(t_{i+1}) = p(t_i) \exp(x(t_i)/\sigma)$, where $\sigma$ is a parameter which controls the size of the updates and provides a measure of liquidity of the market.

We will now present some features of a typical simulation 2000 time steps long with $N = 10,000$, $N_{\text{max}} = 50$, $p = 0.05$, $\alpha = 0.99$, $\beta = 0.003$ and $\phi = 0.01$. The results of our model are compared with the daily closures of Shenzhen Stock Exchange (SZSE) composite index, from January 2nd 1997 to November 22nd 2004, for a total of 1858 working days.

Fig. 2 shows the cumulative distribution $P(r)$ of standardized logarithmic returns $r$ with different time lag $\tau$, i.e., logarithmic returns $r(t) = \log p(t) - \log p(t-\tau)$ detrended by their mean and rescaled by their standard deviation (see Fig. 1). We observe a crossover to the normal distribution with increasing time lag $\tau$. The same happens for empirical financial data. For comparison, the solid line represents the cumulative distribution of the standard normal distribution $N(0, 1)$. From Fig. 2 we can see that the cumulative distribution $P(r)$ exhibits power-law tails with $P(r) \propto r^{-\alpha}$. The value of $\alpha$ is greater than 2, i.e., within the range observed in empirical stock market time series [4].

In Fig. 3 we show a linear–log plot of the probability distribution of normalized returns. The tails of the return distribution of SZSE and our model follow a power-law decay, reflecting the fact that large coherent events, far from the average, are likely to occur with a frequency higher than expected for a random process where the shape would be Gaussian. These large events are related to financial crashes or bubbles of the market.

In Fig. 4, we present the autocorrelation $C(\tau)$ of the absolute returns $|r|$ and of the raw returns $r$ at different time lags $\tau$ for both the model and the SZSE. While the temporal correlation for the return is lost almost immediately in our model, the autocorrelation of the absolute value of returns (volatility) shows the presence of long-range correlations with a very slow power-law decay with exponent $-0.47$, also in agreement with empirical facts [4].

It has been recently noticed that time series of returns in stock markets have multifractal (multiscaling) character. In order to study the multifractal properties of our model we use the generalized Hurst exponent [26], $H(q)$ derived via the $q$-order structure function,

$$S_q(\tau) = \langle |x(t + \tau) - x(t)|^q \rangle_T \propto \tau^{qH(q)}, \quad (7)$$

where $x(t)$ is a stochastic variable over a time interval $T$ and $\tau$ the time delay. If $H(q) = H$ for every $q$ the process is said to be monofractal and $H$ is equivalent to the
Fig. 1. Daily returns of Shenzhen composite index closes from 02/01/1997 to 22/11/2004 (Top). Time series of returns produced by the model (Bottom).

Fig. 2. Log–log plot of the cumulative distribution of returns \( r \). The positive and the negative tails were merged by using absolute returns. The dashed line shows a power law \( r^{-\alpha} \) with exponent \( \alpha = 2.5 \), which is a fit line of absolute returns of the Model with time lag \( \tau = 1 \). For comparison, the solid line gives the complement of the cumulative distribution of the standard normal distribution.
original definition of the Hurst exponent. If the spectrum of $H(q)$ is not constant with $q$ the process is said to be multifractal [10].

In Fig. 5 a comparison is shown between the multifractal spectra of the SZSE and the model obtained from the price time series. It is clear that both processes have a
multifractal structure and the price fluctuations cannot be associated with a simple random walk as in the classical efficient market hypothesis [27].

4. Conclusion

In this paper, we have presented a simple model for the propagation of information and the formation of groups based on belief propagation in financial markets. The beliefs of market leaders spread on a two-dimensional small-world network and those who are influenced by the leaders will be set active and choose the strategies according to all the beliefs they received. Due to shared information, agents do not act independently and make group decisions, a phenomenon known as herding. These group decisions can have a strong impact on the market, modifying drastically the supply and demand equilibrium and, ultimately, the price.

The numerical results of the model are compared with the time series of daily closures of the Shenzhen composite index over a period of about 8 years. The comparison shows that the “stylized facts” of the empirical financial market time series, such as fat-tail distribution of returns, volatility clustering and multifractal property, are also observed in the model.

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References

D.J. Watts, Small Worlds Princeton University Press, Princeton, 1999;